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Type I^+ Helicoidal Surfaces with Prescribed Weighted Mean or Gaussian Curvature in Minkowski Space with Density

Önder Gökmen Yıldız, Selman Hızal and Mahmut Akyiğit

Abstract

In this paper, we construct a helicoidal surface of type I^+ with prescribed weighted mean curvature and Gaussian curvature in the Minkowski 3−space $\mathbb{R}^3_1$ with a positive density function. We get a result for minimal case. Also, we give examples of a helicoidal surface with weighted mean curvature and Gaussian curvature.

1 Introduction

It is well known that a helicoidal surface is a generalization of a rotation surface and has been studied in $\mathbb{R}^3$ as well as in the other spaces with special conditions. Firstly, a helicoidal surface in Euclidean 3−space has been studied by Do Carmo [4]. Then, the helicoidal surfaces with prescribed mean and Gaussian curvature in Minkowski 3−space has been studied by Beneki et. al. [2] and Ji et. al. [7]. Also, the helicoidal surfaces with prescribed mean and Gaussian curvature in Euclidean 3-space with density have been studied by Won Yoon et. al. [15]. One can refer to [1, 5, 8] for more details.

Recently, the studies in Riemannian manifolds with density have arisen. A manifold with a positive density function e^{φ} used to weight the volume and the hypersurface area. For more details on manifolds with density, see [6, 9, 10, 11, 12, 13, 14].

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In Minkowski 3−space with density e^{φ}, the weighted mean curvature is given with

$$H_{\varphi} = H - \frac{1}{2} \langle N, \bigtriangledown\varphi \rangle$$

where H is the mean curvature of the surface, N is the unit normal vector of the surface and $\bigtriangledown\varphi$ is the gradient vector of φ [13]. If $H_{\varphi} = 0$ then the surface is called weighted minimal surface. The weighted Gaussian curvature with density e^{φ} is

$$G_{\varphi} = G - \triangle\varphi$$

where G is the Gaussian curvature of the surface and $\triangle$ is the Laplacian operator [3].

In this paper, we study helicoidal surfaces in Minkowski 3−space $\mathbb{R}^3_1$ with density e^{φ}, where $\varphi = -x_1^2 - x_2^2$. Firstly, we consider helicoidal surfaces of type I^+, defined in [2]. Then, we construct a helicoidal surface of type I^+ with prescribed weighted mean and Gaussian curvature. Finally, we give examples to illustrate our result.

2 Preliminaries

Minkowski 3−space $\mathbb{R}^3_1$ is the real vector 3−space $\mathbb{R}^3$ provided with the Lorentzian metric

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$$

where $(x_1, x_{2,} x_3)$ is a rectangular coordinate system of $\mathbb{R}^3_1$.

For a given plane curve and an axis in the plane in $\mathbb{R}^3_1$, a helicoidal surface can be constructed by the plane curve under helicoidal motions $g_t : \mathbb{R}^3_1 \to \mathbb{R}^3_1, t \in \mathbb{R}$ around the axis. So, a helicoidal surface is non-degenerate and invariant under $g_t, t \in \mathbb{R}$ one-parameter subgroup of rigid motions in $\mathbb{R}^3_1$. There exist four kinds of helicoidal surfaces in $\mathbb{R}^3_1$ which are defined by Beneki et. al. [2] and these are called type I, type II, type III , type IV. In this study, type I^+ is considered which has the spacelike axis of revolution and the profile curve lies in the x_2x_3−plane. In addition, a helicoidal surface is called type I^+ since discriminant of the first fundamental form is positive [2].

Let γ be a C^2−curve on x_2x_3−plane, of type $\gamma(v) = (0, v, g(v)), v \in I$ for an open interval $I \subset \mathbb{R} - \{0\}$. By using helicoidal motion on γ, we can obtain the helicoidal surface as

$$r(v,u) = \begin{bmatrix} \cosh u & \sinh u & 0 \\ \sinh u & \cosh u & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ v \\ g(v) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ cu \end{bmatrix} \tag{2.1}$$

with x_3-axis and a pitch $h \in \mathbb{R}$, so the parametric equation can be given in the form

$$r(v,u) = (v \sinh u, v \cosh u, g(v) + cu). \tag{2.2}$$

We can calculate the mean curvature H, the Gaussian curvature G and the unit normal of surface as

$$H = \frac{\left(1+g'^2\right) v^2 g' - \left(c^2 - v^2\right) v g'' - 2c^2 g'}{2\left[c^2 - v^2\left(1+g'^2\right)\right]^{3/2}}$$

$$G = -\frac{v^3 g' g'' + c^2}{\left[c^2 - v^2\left(1+g'^2\right)\right]^2}$$

and

$$N = \frac{1}{\sqrt{w}}\left(g'(v) v \sinh u - c\cosh u, g'(v) v \cosh u - c \sinh u, -v\right)$$

respectively, where $c^2 - v^2\left(1+g'^2\right) > 0$ and $w = c^2 - v^2\left(1+g'^2\right)$ [2]. We assume that M is a surface in $\mathbb{R}^3_1$ with density e^{φ}, where $\varphi = -x_1^2 - x_2^2$. By considering density function, we can calculate the weighted mean curvature H_φ and the weighted Gaussian curvature G_φ as

$$H_\varphi = \frac{\left(v^2 - c^2\right) v g'' + \left(v^2 - 2v^4\right) g'^3 + \left(v^2 - 2c^2 - 2v^4 + 2c^2 v^2\right) g'}{2 w^{3/2}} \tag{2.3}$$

and

$$G_\varphi = -\frac{v^3 g' g'' + c^2}{w^2} + 4, \tag{2.4}$$

respectively.

3 Helicoidal surfaces with prescribed weighted mean or Gaussian curvature

Theorem 3.1. *Let $\gamma(v) = (0, v, g(v))$ be a profile curve of the helicoidal surface given with $r(v,u) = (v\sinh u, v\cosh u, g(v) + cu)$ in $\mathbb{R}^3_1$ with density $e^{-x_1^2 - x_2^2}$ and $H_\varphi(v)$ be the weighted mean curvature. Then, there exists $2-$parameter family of helicoidal surface given by the curves*

$$\gamma\left(v, H_\varphi(v), c, c_1, c_2\right) = \left(v, 0, \mp \int \frac{e^{v^2}\sqrt{c^2 - v^2}\left(-2\int v e^{-v^2} H_\varphi dv + c_1\right)}{v\sqrt{v^2 + e^{2v^2}\left(-2\int v e^{-v^2} H_\varphi dv + c_1\right)}} + c_2\right).$$

Conversely, for a given smooth function $H_\varphi(v)$, one can obtain the two-parameter family of curves $\gamma\left(v, H_\varphi(v), c, c_1, c_2\right)$ being the two-parameter family of helicoidal surfaces, accepting $H_\varphi(v)$ as the weighted mean curvature c as a pitch.

Proof. Let's solve the equation (2.3) which is second-order nonlinear ordinary differential equation. If we apply $A = \frac{g'(v)}{\sqrt{w}}$ into the equation, then we get

$$H_\varphi = -\frac{v}{2}A' + \left(-1 + v^2\right) A. \tag{3.1}$$

The equation (3.1) becomes a first-order linear ordinary differential equation with respect to A and we rewrite the equation as follows

$$A' + \left(\frac{2}{v} - 2v\right) A = -\frac{2}{v} H_\varphi. \tag{3.2}$$

We get the general solution of the equation (3.2) as

$$A = \frac{e^{v^2}}{v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right), \tag{3.3}$$

where $c_1 \in \mathbb{R}$. By using $A = \frac{g'(v)}{\sqrt{w}}$ and the equation (3.3), we obtain

$$\left[v^2 + e^{2v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right)^2\right] g'^2(v) = \frac{\left(c^2 - v^2\right)}{v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right)^2 \tag{3.4}$$

and integrate the equation (3.4), we get

$$g(v) = \mp \int \frac{e^{v^2}\sqrt{c^2 - v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right)}{v\sqrt{v^2 + e^{2v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right)^2}} dv + c_2 \tag{3.5}$$

where $c_2 \in \mathbb{R}$.

On the contrary, for a given constant $c \in \mathbb{R} - \{0\}$, a real-valued smooth function $H_\varphi(v)$ defined on an open interval $I \subset \mathbb{R}^+$ and an arbitrary $v_0 \in I$, there exists an open subinterval $v_0 \in I' \subset I$ and an open interval $J \subset \mathbb{R}$ which contains

$$c_1' = \left(2\int ve^{-v^2} H_\varphi dv\right)(v_0)$$

such that

$$F(v, c_1) = v^2 + e^{2v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right)^2 > 0$$

for arbitrary (v, c_1). Since $F(v_0, c_1') = v^2 > 0$ and F is continuous, F is positive on $I' \times J \subset \mathbb{R}^2$. Thus, 2-parameter family of the curves can be given as

$$\gamma(v, H_\varphi(v), c, c_1, c_2) = \left(0, v, \mp \int \frac{e^{v^2}\sqrt{c^2 - v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right)}{v\sqrt{v^2 + e^{2v^2}\left(-2\int ve^{-v^2} H_\varphi dv + c_1\right)^2}} dv + c_2\right).$$

□

Corollary 3.2. *Let M be a minimal helicoidal surface in $\mathbb{R}_1^3$ with density $e^{-x_1^2-x_2^2}$. Then M is an open part of either a helicoid or a surface parametrized by*

$$r(v,u)=\left(v\sinh u, v\cosh u, \mp\int \frac{c_1 e^{v^2}\sqrt{c^2-v^2}}{v\sqrt{v^2+c_1^2 e^{2v^2}}}dv+c_2+cu\right)$$

where $c_1, c_2 \in \mathbb{R}$.

Example 3.3. Consider a helicoidal surface with the weighted mean curvature

$$H_\varphi(v)=\frac{2v^2-1}{v}$$

and the pitch $c=1$ in $\mathbb{R}_1^3$ with density $e^{-x_1^2-x_2^2}$. By considering the equation (3.5), we get $\gamma(v)$, hence we obtain the parametrization of the surface as follows

$$r(v,u)=\left(v\sinh u, v\cosh u, -\frac{\sqrt{1-v^2}+\ln(v)-\ln\left(1+\sqrt{1-v^2}\right)}{\sqrt{2}}+u\right)$$

and the figure of the domain

$$\begin{cases} 0<v<1 \\ -3<u<3 \end{cases}$$

is given in Figure 1.

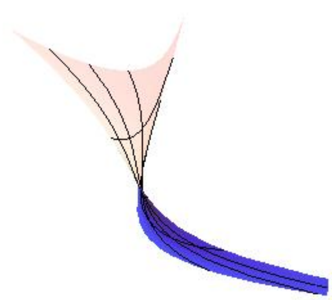

Figure 1: The helicoidal surface with the weighted mean curvature

The difference between H and H_φ of the helicoidal surface with density can be seen in Figure 2.

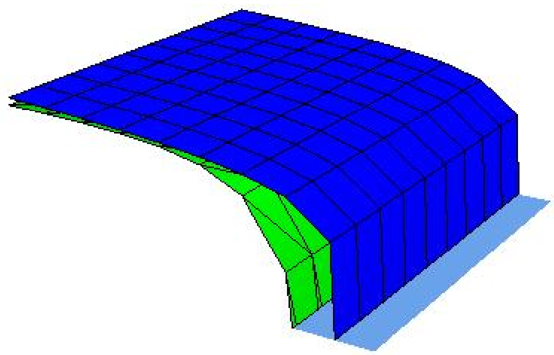

Figure 2: H (Green) and H_φ (Blue)

Theorem 3.4. *Let $\gamma(v) = (0, v, g(v))$ be a profile curve of the helicoidal surface given with $r(v,u) = (v\sinh u, v\cosh u, g(v) + cu)$ in $\mathbb{R}^3_1$ with density $e^{-x^2-y^2}$ and $G_\varphi(v)$ be the weighted Gaussian curvature at $(0, v, g(v))$. Then, there exist the $2-$parameter family of the helicoidal surface given by the curves*

$$\gamma(v, G_\varphi(v), c, c_1, c_2) = \left(0, v, \mp\int \frac{1}{v}\left[\frac{(c^2-v^2)\left(4v^2+2\int vG_\varphi dv + c_1\right)+c^2}{1+4v^2+2\int vG_\varphi dv + c_1}\right]^{\frac{1}{2}} dv + c_2\right)$$

here, c_1 and c_2 are constants. Conversely, for a given smooth function $G_\varphi(v)$, one can obtain the $2-$parameter family of curves $\gamma(v, G_\varphi(v), c_1, c_2)$ being the $2-$parameter family of helicoidal surfaces, accepting $G_\varphi(v)$ as the weighted Gaussian curvature c as a pitch.

Proof. Let's solve the equation (2.4), which is second-order nonlinear ordinary differantial equation. If we substitute

$$B = \frac{v^2 g'^2 - c^2}{c^2 - v^2\left(1+g'^2\right)} \tag{3.6}$$

into the equation (3.6), then we obtain

$$G_\varphi = \frac{1}{2v}B' + 4$$

that is,

$$B' = 2vG_\varphi - 8v. \tag{3.7}$$

We get the general solution of the equation (3.7) as

$$B = -4v^2 + 2\int vG_\varphi dv + c_1 \tag{3.8}$$

where $c_1 \in \mathbb{R}$. By using the equation (3.6) and the equation (3.8), we get

$$v^2\left(1 - 4v^2 + 2\int vG_\varphi dv + c_1\right){g'}^2 = \left(c^2 - v^2\right)\left(-4v^2 + 2\int vG_\varphi dv + c_1\right) + c^2. \tag{3.9}$$

From (3.9) we obtain

$$g(v) = \mp\int \frac{1}{v}\left[\frac{\left(c^2 - v^2\right)\left(-4v^2 + 2\int vG_\varphi dv + c_1\right) + c^2}{1 - 4v^2 + 2\int vG_\varphi dv + c_1}\right]^{\frac{1}{2}} dv + c_2 \tag{3.10}$$

where $c_2 \in \mathbb{R}$.

On the contrary, for a given $h \in \mathbb{R}$ and a smooth function $G_\varphi(v)$ defined on an open interval $I \subset \mathbb{R}^+$. An arbitrary $v_0 \in I$, there exists an open subinterval $v_0 \in I' \subset I$ and an open interval $J \subset \mathbb{R}$ which contains

$$c_1' = -\left(-4v^2 + 2\int vG_\varphi dv\right)(v_0)$$

such that

$$F(v, c_1) = 1 - 4v^2 + 2\int vG_\varphi dv > 0$$

is defined on $I' \times J$ and it is easily seen F is $I' \times J \subset \mathbb{R}^2$. Thus, 2−parameter family of the curves can be given as

$$\gamma(v, G_\varphi(v), c, c_1, c_2) = \left(v, 0, \mp\int \frac{1}{v}\left[\frac{\left(c^2 - v^2\right)\left(4v^2 + 2\int vG_\varphi dv + c_1\right) + c^2}{1 + 4v^2 + 2\int vG_\varphi dv + c_1}\right]^{\frac{1}{2}} dv + c_2\right)$$

where $(v, c_1) \in I' \times J, c_2 \in \mathbb{R}, c \in \mathbb{R}$ and G_φ is smooth function. □

Example 3.5. Consider a helicoidal surface with the weighted Gaussian curvature

$$G_\varphi(v) = -\frac{\left(1 + v^2\right)\left(1 + 3v^2 + v^4 + v^6\right)}{\left(-1 + v^4 + v^6\right)^2} + 4$$

in R_1^3 with density $e^{-x_1^2 - x_2^2}$. By using the equation (3.10), we obtain $g(v) = \arctan v$ for $c = 1, c_1 = 0, c_2 = 0$ and we obtain the parametrization of the surface as follows

$$r(v, u) = (v \sinh u, v \cosh u, \arctan v + u).$$

The figure of the surface of the domain

$$\begin{cases} -5 < v < 5 \\ -1 < u < 1 \end{cases}$$

is given in Figure 3. The difference between G and G_φ of the helicoidal surface with density can be seen in Figure 4.

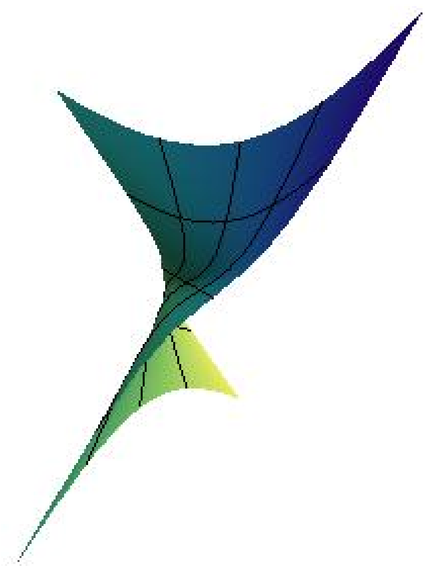

Figure 3: The helicoidal surface with the weighted Gaussian curvature

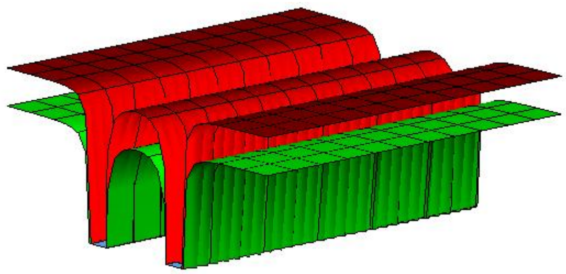

Figure 4: G (Green) and G_φ (Red)

4 Conclusion and future work

In this paper, using density $e^{-x_1^2-x_2^2}$, we construct a helicoidal surface of type I^+ with prescribed weighted mean curvature and Gaussian curvature in the Minkowski 3−space $\mathbb{R}_1^3$ with a positive density function. By using different density functions, different helicoidal surfaces can be obtained in a manifold with density. Besides, in a similar manner to in this paper one can study helicoidal surface for type II, III and IV .

References

[1] C. Baikoussis, T. Koufogiorgos, Helicoidal surfaces with prescribed mean or Gaussian curvature, Journal of Geometry **63**.(1) (1998) 25-29.

[2] C. C. Beneki, G. Kaimakamis, B. J. Papantoniou, Helicoidal surfaces in three-dimensional Minkowski space, Journal of Mathematical Analysis and Applications **275**.(2) (2002) 586-614.

[3] I. Corwin, N. Hoffman, S. Hurder, V. eum, Y. Xu, Differential geometry of manifolds with density. Rose-Hulman Undergrad. Math. J. **7** (2006) 1-15.

[4] M. P. Do Carmo, M. Dajczer, Helicoidal surfaces with constant mean curvature, Tohoku Mathematical Journal, Second Series **34**.(3) (1982) 425-435.

[5] C. H. Delaunay, Sur la surface de révolution dont la courbure moyenne est constante, Journal de mathématiques pures et appliquées (1841) 309-314.

[6] D. T. Hıeu, N. M. Hoang, Ruled minimal surfaces in $\mathbb{R}^3$ with density e^z, Pacific Journal of Mathematics **243**.(2) (2009) 277-285.

[7] F. Ji, Z. H. Hou, A kind of helicoidal surfaces in 3-dimensional Minkowski space, Journal of mathematical analysis and applications **304**.(2) (2005) 632-643.

[8] F. Ji, Z. H. Hou, Helicoidal surfaces under the cubic screw motion in Minkowski 3-space, Journal of mathematical analysis and applications **318**.(2) (2006) 634-647.

[9] F. Morgan, Geometric measure theory: a beginner's guide. Academic press (2016).

[10] F. Morgan, Manifolds with density, Notices of the AMS **52**.(8) (2005) 853-858.

[11] F. Morgan, Manifolds with density and Perelman's Proof of the Poincaré conjecture, American Mathematical Monthly **116**.(2) (2009) 134-142.

[12] F. Morgan, Myers' theorem with density, Kodai Mathematical Journal **29**.(3) (2006) 455-461.

[13] P. Rayón, M. Gromov, Isoperimetry of waists and concentration of maps, Geometric and functional analysis **13**.(1) (2003) 178-215.

[14] C. Rosales, A. Canete, V, Bayle, F. Morgan, On the isoperimetric problem in Euclidean space with density, Calculus of Variations and Partial Differential Equations **31**.(1) (2008) 27-46.

[15] D. W. Yoon, D. S. Kim, Y. H. Kim, J. W. Lee, Constructions of Helicoidal Surfaces in Euclidean Space with Density, Symmetry **9**.(9) (2017) 173-182.

Önder Gökmen Yıldız,
Department of Mathematics,
Faculty of Arts and Science,
Bilecik Şeyh Edebali University,
Gülümbe Campus, 11230 Bilecik, Turkey.
Email: ogokmen.yildiz@bilecik.edu.tr

Selman Hızal,
Distance Education Research And Application Centre,
Sakarya University,
Esentepe Campus, 54187 Sakarya, Turkey.
Email: shizal@sakarya.edu.tr

Mahmut Akyiğit,
Department of Mathematics,
Faculty of Arts and Science,
Sakarya University,
Esentepe Campus, 54187 Sakarya, Turkey.
Email: makyigit@sakarya.edu.tr